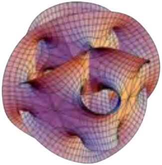
Exploring the Differences and Connections Between Two Generalizations of the Penrose Polynomial via Knot Theory



Minxiao Wu[1]

[1]Dpartment of Mathematics
Ohio State University

Knots and Graphs Working Group, July 2026

Table of Contents

1	A Quick Review
2	Aigner Blow-Up Construction
3	Quantitatively Study

Outline

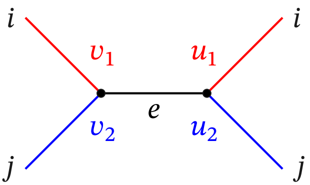
1 A Quick Review

2 Aigner Blow-Up Construction

3 Quantitatively Study

Tait Coloring

Given a cubic graph G and perfect matching M, a Tait-n coloring of such a pair (G, M) is an assignment of colors $\{1, 2, \ldots, n\}$ to edges of G not in M, which satisfies that for any edge $e \in M$, the two pairs of edges, $\{v_1, v_2\}$ and $\{u_1, u_2\}$, adjacent to it at its respective endpoints share the same coloring pattern, say $\{i, j\}$, where $i \neq j$.



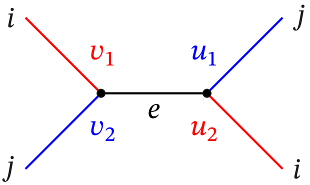


Figure 1. Tait n-Coloring Condition

PK Polynomial

Definition 1 (Penrose Polynomial PK)

Let G be a cubic graph embedded in a surface together with a perfect matching M. The Penrose-Kauffman (PK) Polynomial of (G, M) is the unique integral polynomial $PK(x)$ with the property that for any integer $n \geq 1$, $PK(n)$ is the number of Tait n-colorings of (G, M). ♣

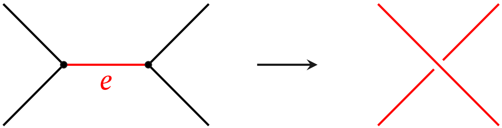


Figure 2. The Corresponding Link Diagram D

A Computation Given by Chromatic Polynomial

Theorem 1

Let G be a cubic graph with perfect matching M embedded in a surface. The PK polynomial of (G, M) satisfies that

$$PK_{(G,M)}(q) = \sum_{S} \chi(G_S, q). \tag{1}$$

- Note that for any Tait n-coloring of the pair (G, M), by connecting two edges in the same color that adjacent to the edge $e \in M$ we get a unique n-coloring of a colorable state.
- Conversely, any n-coloring of a colorable state can uniquely correspond to a Tait n-coloring of (G, M).

Tait-n Coloring and Colorable State

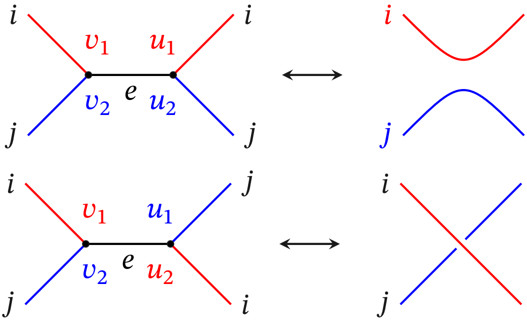
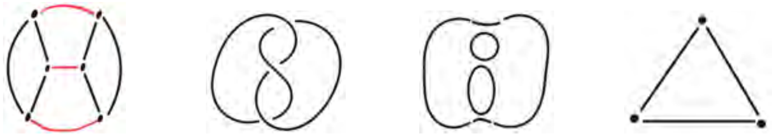


Figure 3. The Transformation of a Tait n-coloring and a n-coloring of a Colorable State at a Crossing

An Example for the Calculation of PK

By Fig.4, the PK polynomial of the left hand trefoil knot is

$$PK_D(q) = \chi(G_S, q) = q(q-1)(q-2). \tag{2}$$



(a) (G, M) **(b)** $D(G, M)$ **(c)** Colorable State **(d)** G_S

Figure 4. The Calculation of the PK Polynomial

Penrose Polynomial

Definition 2 (Penrose Polynomial)

Let G be an embedded graph with canonically checkerboard coloured medial graph G_m, let $S(G_m)$ be the set of states of G_m. Then the Penrose polynomial is defined by

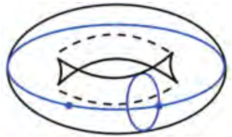
$$P_G(q) := \sum_{s \in S(G_m)} \omega_p(s) q^{c(s)} = \sum_{s \in \mathscr{P}(G_m)} (-1)^{cr(s)} q^{c(s)}, \quad (3)$$

where $\omega_p(s)$ denotes the graph state weights of the medial weight system $W_p(G_m)$ defined by $(1, 0, -1)$, where $c(s)$ is the number of components in the graph state and $cr(s)$ is the number of crossing vertex states in the graph state s. ♣

An Example of the Calculation of the Penrose

By analyzing each Penrose state individually, the Penrose polynomial of an embedded graph G shown in Fig.5 is

$$P_G(q) = \sum_{s \in \mathscr{P}(G_m)} (-1)^{cr(s)} q^{c(s)} = 2q^2 - 2q. \tag{4}$$



(a) The Embedded Graph G

(b) The Medial Graph G_m

Figure 5. An Embedded Graph G and its Medial Graph G_m

Theorem 2 (Thm.4.24 in [1])

If G is an embedded graph and $n \in \mathbb{N}$, then

$$P_G(n) = \sum_{s \in \mathcal{A}_n(G_m)} (-1)^{cr(s)}, \tag{5}$$

where $\mathcal{A}_n(G_m)$ is the set of all admissible n-valuation of the medial graph G_m.

♡

Theorem 3

If G is an embedded graph. Then

$$P_G(q) = \sum_{A \subseteq E(G)} (-1)^{|A|} \chi\left[\left(G^{\tau(A)}\right)^*, q\right]. \tag{6}$$

♡

Outline

1 A Quick Review

2 Aigner Blow-Up Construction

3 Quantitatively Study

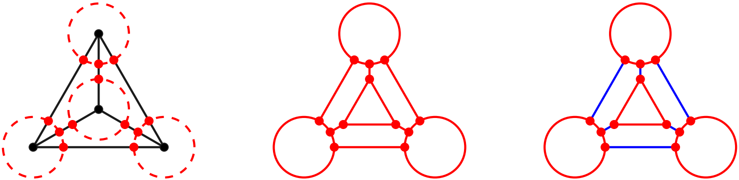
The Blow-Up Graph

For a plane graph E we construct its Aigner's blow-up graph $\hat{E}$ by the following steps:

1. Draw a sufficiently small circle centered at each vertex of $\hat{E}$; the intersection points of these circles with the half-edges of E emanating from the respective vertices constitute the vertices of the blow-up graph $\hat{E}$.
2. We draw an edge of $\hat{E}$ between any two of its vertices if and only if they are in one of the following situations:
   (a) The half-edges they originally occupied are adjacent (i.e., they originate from or converge at the same vertex of E).
   (b) They are on a same edge of E.

From this, one could easily get that the blow-up construction of a plane graph is cubic and all the original edge form a perfect matching.

The Blow-Up Graph



(a) Embedded Graph E **(b)** Blow-Up Graph $\hat{E}$ **(c)** Perfect Matching $\hat{M}$

Figure 6. An Example of the Aigner Blow-Up Construction

A Tait n-coloring of $(\hat{G}, \hat{M})$ is equivalent to an admissible n-valuation of the checkerboard coloring medial graph G_m. Therefore by Jeager's theorem we know that $PK_{(\hat{G},\hat{M})}(q) = P_G(q)$.

Alternating Property

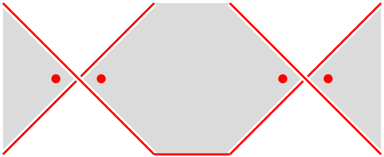


Figure 7. An Example of the Alternativity of the Link Diagram

We say a PK polynomial of a pair (G, M) arising from Aigner's construction if there is a plane graph E whose Aigner's blow-up $(\hat{E}, \hat{M}) = (G, M)$. Not all PK polynomial arising from Aigner's construction. Some PK polynomial of nonalternating link diagram can be different from those polynomial of alternating link diagram.

A PK not arise from Aigner Construction

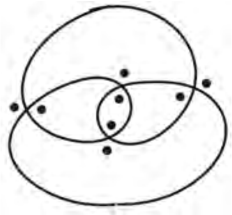


Figure 8. An Example of a *PK* polynomial that is not arised from Aigner Construction

The non-alternating diagram D shown in Fig.8 has three colorable states, each with three components. The PK polynomial is $PK_D(q) = 3q^3 - 8q^2 + 5q$. By the proposition 8.8 in [2] this polynomial can't be arised from an Aigner's construction.

A Conjecture

Conjecture 1

In the planar case for any alternating link diagram D there exists a non-alternating link diagram D' such that $PK_D(q) = PK_{D'}(q)$.

- From Corollary 4.21 of [1] the Penrose polynomial of a plane graph G, $P_G(q) = 0$ if and only if G has a bridge.
- By Proposition 3.7 in [2] we know that $PK = 0$ if G can be drawn in the form of (a).

Outline

1 A Quick Review

2 Aigner Blow-Up Construction

3 Quantitatively Study

Proof-based Environment

Definition 3 (Natural Alternation Complexity)

For a plane link diagram $D = D(G, M)$, we define its natral non-alternating complexity $\alpha(D)$ as the minimum number of crossings whose over/under information must be changed to transform it into an alternating link diagram without altering its projection. ♣

prime non-alternating knot diagram has a same PK polynomial with a prime alternating knot diagram. $D_A = \overline{8_2}$ (8 crossings), $D_N = \overline{9_{49}}$ (9 crossings), They both have 32 colorable states and have the same PK polynomial

$$32q(q-1)(q-2) \tag{7}$$

Bibliography

[1] Joanna A Ellis-Monaghan and Iain Moffatt, *Graphs on surfaces: dualities, polynomials, and knots*, Vol. 84, Springer, 2013.

[2] Louis H Kauffman, Daniel S Silver, and Susan G Williams, *The penrose-kauffman polynomial*, arXiv preprint arXiv:2604.16635 (2026).

Thanks for your listening!

You are welcome to join the discussion.